General Envelope Theorems for Multidimensional Type Spaces

Marcelo de Carvalho Griebeler\(^\dagger\)  Jorge Paulo de Araújo\(^\ddagger\)

IMPA  FCE/UFRGS

February 10, 2010

Abstract

The main objective of this paper is to expand the analysis of Milgrom and Segal (2002) and to generalize their general Envelope Theorem to multidimensional type spaces. In order to do so, we use the Berge’s Maximum Theorem and therefore we need to insert the assumption of compactness of the choice set. However, we are still allowing that the choice rule (mechanism) to be discontinuous. We also identify conditions for the value function to be absolutely continuous and show that its integral representation is valid for multidimensional type spaces. These results allow a large number of applications in multi-agent settings. In particular, models of Public Economics, like those of public goods supply and optimal taxation, and auction theory, are suitable example of our findings’ applications.

Keywords: mechanism design; Envelope Theorem; discontinuous mechanisms.

JEL classification: C60; C72; D86.

1 Introduction

It is common knowledge that Envelope Theorems have several applications in Economics. In particular, its use for concave optimization problems in demand theory is well known. It allows to analyze the effects of changing prices, income, and technology on the welfare and profits of consumers and firms. As the Economics of Information has been developed, another application of this theorem has been raised. Since Mirrlees (1971), it has been used in mechanism design theory with continuous type spaces. Results like the Revenue Equivalence Theorem for auctions and the Myerson-Satterthwaite inefficiency theorem are examples of its application. Nonetheless, in order to apply the traditional theorem is necessary that the choice set has a convex and topological structure.

\(^\dagger\)Address: Instituto Nacional de Matemática Pura e Aplicada, Estrada Dona Castorina, 110, Jardim Botânico, Rio de Janeiro - RJ, Brazil. ZIP code: 22460-320. E-mail: marcelog@impa.br.

\(^\ddagger\)Address: Universidade Federal do Rio Grande do Sul, Faculdade de Ciências Econômicas, Av. João Pessoa, 52 sala 29J - 2\(^{o}\) andar, Centro, Porto Alegre-RS, Brazil. ZIP code: 90040-000. E-mail: 00006283@ufrgs.br.
Because of this limitation, traditional models of mechanism design frequently assume that the choice rules are piecewise continuously differentiable or that the choice sets have a standard structure. Indeed, the seminal papers in this area introduced\textsuperscript{1} this assumption as a technical simplification, and later studies, including the most recent ones\textsuperscript{2}, have followed this fashion. This simplification has allowed the great development of applications in a variety of areas (Public Economics, price discrimination, Labor Economics). However, several examples in many different fields of Economics have showed that that assumption may not be satisfactory. Laffont and Tirole (1993) and Myerson (1991), for example, present discontinuous optimal mechanisms in Regulation Economics and bilateral trading, respectively.

More recently, two large literatures have been developed concerning mechanism design theory. The first one has maintained the assumption of differentiability and is formed by, in its most, applied studies. However, papers like Williams (1999) have made advances in theory without keeping out the continuity of choice rule. Its findings are concerning characterization of efficient mechanisms and are based on the Envelope Theorem and in the integral representation of the utility. Therefore, this literature has followed traditional models, in the sense of technical assumptions, and its merit is to increase the scope of economic applications.

The second kind of recent literature in mechanism design has studied mechanisms making no assumption about choice rule. Milgrom and Segal (2002) is its main representant, because these authors present very general results in theory. The most important finding of this study is a generalized Envelope Theorem to arbitrary choice sets. Under a few assumptions about the objective function, mainly absolute continuity, it is showed that the Envelope Theorem holds even for discontinuous choice rules. This is a powerful result in mechanism design theory, because of the large importance of this theorem in solving a large variety of applications. However, all analysis of Milgron and Segal (2002) only applies to single-agent settings and its generalization to multi-agents is not straightforward. Their approach may be useful in a few applications (bilateral trading and regulation are examples) but it is not enough to cover all mechanism design theory. In the case of discontinuity in the classic models of public good supply and price discrimination, for instance, this framework does not work.

Other studies have developed general forms of the "payoff equivalence", whose the Revenue-Equivalence Theorem is a particular case. Although this literature has found results for multidimensional type spaces, its findings only apply to settings where the agent’s utility is separable in the payment (or transfer) function. Krishna and Maenner (2001), for example, show that the integral representation (payoff equivalence) holds under one of two assumptions is satisfied. The first one is convexity of the type spaces and of the utility function; and the second establishes that the mechanism is regular Lipschitzian and the utility is regular Lipschitzian and monotonically increasing functions in all its arguments.

Chung and Olszewski (2007) also establishes conditions for the revenue equivalence to be valid in multi-agent settings. When the set of social alternatives consists of probability distributions over a finite set, their findings generalize all existing revenue equivalence theorems. Their main assumption

\textsuperscript{1}See mainly Laffont and Maskin (1980) and Guesnerie and Laffont (1984).
\textsuperscript{2}See Williams (1999), for instance.
requires that the mechanism’s domain to be boundedly gridwise connected. It is shown that this sufficient condition is stronger than connectedness but weaker than smooth arcwise connectedness. However, this paper has the same limitation of Krishna and Manner (2001), in the sense of using quasi-linear utilities and focusing only in the revenue equivalence.

The main objective of this paper is to expand the analysis of Milgrom and Segal (2002) and to generalize their results to multidimensional type spaces. In order to do so, we use the Berge’s Maximum Theorem and therefore we need to insert the assumption of compactness of the choice set. However, we are still allowing that the choice rule to be discontinuous. It may seem strong to impose this assumption, but we argue that in mechanisms design is not common that choice sets be open or unbounded. We also identify conditions for the value function to be absolutely continuous and show that its integral representation is valid for multidimensional type spaces. Our results are applicable specially in Public Economics (models of public good supply and income taxation) and auction theory.

Besides this brief introduction and literature review, our paper divides in more three sections. We start discussing the limitations caused by non-differentiability in mechanism design problems in the section 2. We show that the Envelope Theorem actually might not hold under this assumption. Section 3 generalizes the findings of Milgrom and Segal (2000) for multidimensional type spaces and argue the importance of imposing the assumption of compactness. Section 4 concludes.

2 Limitations Imposed by Non-Differentiability

In order to see the problems caused by non-differentiability in mechanism design models, let \( u_0(\cdot) \) be the principal’s utility function and \( u_i(\cdot) \) be the agent’s utility, with \( i = 1, ..., k \). Let also \( t_i \) represent the agent \( i \)’s type, where \( t_i \in I_i \). We have then \( t = (t_1, ..., t_k) \in I = I_1 \times ... \times I_k \). For the sake of simplicity, we assume a standard setting, with a bidimensional choice rule \( x_i(t) = (q_i(t), p_i(t)) \), but our argument might be easily generalized for multidimensional \( q_i(\cdot) \). In this context, \( X \) is the choice set and then \( x_i \in X \). Moreover, let \( M_i \) denote the message space for each agent \( i \), such that \( M = M_1 \times ... \times M_k \), and let \( \mu = (\mu_1, ..., \mu_k) \) be the vector of all messages sent by the agents. Thus, we have \( u_0(q(\mu), p(\mu), t) \) and \( u_i(q(\mu), p(\mu), t) \), for each \( i \), and the function \( \mu : I \to M \) gives opportunity to agents behave strategically.

The structure above is very general and might model a variety of economic problems (price discrimination, regulation, income tax). In particular, in an auction model, \( q(\cdot) \) would be the bidder’s probability of getting the good and \( p(\cdot) \) would be the payment function. Note that we do not assume any functional form for utilities and let \( I, X \) and \( M \) be arbitrary. Therefore, our general model is formed by a principal and \( k \) agents, \( i = 1, ..., k \). The principal wishes to maximize her (expected, in a bayesian setting) utility in choosing the pair of functions \( (q_i(\cdot), p_i(\cdot)) \), for each \( i \). Yet, \( t_i \) is private value of agent \( i \), such that the principal only observes \( \mu_i \). We also assume that she knows the types’ probability function. Thus, it is necessary to give incentives to agents behave strategically.

It is also possible to see the principal’s problem as choosing the mechanism that implements a given social choice. Of course, there exists a large number os pairs \( (q_i(\cdot), p_i(\cdot)) \), for each \( i \), which
do so, and the task of finding the best one for the principal may be difficult. Fortunately, we may restrict our attention to truthful direct revelation mechanisms in solving the model. This is allowed by the Revelation Principle\textsuperscript{3}. It implies that $M = I$. In other words, direct revelation mechanisms are equivalent to ask to agents to say their type. Consequently, we work directly with $q(t)$ and $p(t)$. In addition, in being truthful, the mechanism should satisfy the incentive-compability (IC) constraint:

$$u_i(q(t), p(t), t) \leq u_i(q(t), p(t), t), \text{ for all } i \text{ and all } (t, \hat{t}),$$

where $\hat{t}$ is the agent reported type. Thus, the value functions of the problem are the indirect utilities $V_i(t) \equiv \max_{q, p} u_i(q(t), p(t), t)$, for all $i$.

If we assume implementation in bayesian equilibrium, then the principal maximizes her expected utility subject to the agent’s individual-rationality (IR) and incentive-compability (IC) constraints:

$$\max_{\{q(\cdot), p(\cdot)\}_{t=1,\ldots,k}} E_t u_0(q(t), p(t), t)$$

subject to

$$u_i(q(t), p(t), t) \geq u_0, \text{ for all } i \text{ and all } t,$$

$$u_i(q(t), p(t), t) \geq u_i(q(\hat{t}), p(\hat{t}), t), \text{ for all } i \text{ and all } (t, \hat{t}).$$

It is important to impose few more standard assumptions in the problem above in order to guarantee existence and uniqueness of solution. First, we assume that reservation utility $u_0$ is independent of type and, for notational simplicity, we normalize $u_0 = 0$. Moreover, utility is increasing in the own type of the agent, that is, $\frac{\partial u_i}{\partial t_i} > 0$, for all $i$. Finally, let the Spence-Mirrlees (or constant signal) condition\textsuperscript{4} hold, such that either

$$\frac{\partial}{\partial t_j} \left( \frac{\partial u_i}{\partial q} \right) > 0, \text{ for all } i \text{ and all } j; (CS^+),$$

or

$$\frac{\partial}{\partial t_j} \left( \frac{\partial u_i}{\partial q} \right) < 0, \text{ for all } i \text{ and all } j; (CS^-).$$

As the agent is also an utility maximizer, he should choose his reported type $\hat{t}$ which gives him the maximum value. We might show that the first and the second order conditions of agent’s maximization problem, $\frac{\partial u_i}{\partial t_i} = 0$ and $\frac{\partial^2 u_i}{\partial t_i^2} \leq 0$, respectively, yield a system of differential equations which replaces the IC and IR constraints in the problem above\textsuperscript{5}. Further, the expectation operator in the objective function indicates that it is in an integral form, which is weighted by the types’ density function. Accordingly, the (2), (3) and (4) become a standard dynamic optimization problem.

In this formulation, the model may be solved by standard techniques of optimal control. Here,\textsuperscript{3}See Fundenberg and Tirole (1995) and Myerson (1991) for straightforward proofs of the principle in a bayesian equilibrium setting.

\textsuperscript{4}This condition asserts that the agent’s type affects his marginal rate of substitution in a systematic way. Moreover, it is used in almost all applications. We will restrict our attention to the cases when $CS^+$ holds for all $i$.

\textsuperscript{5}For a complete treatment of this subject, see Laffont and Maskin (1980).
we see one of the limitations imposed by non-differentiability in the choice rule $x_i(\cdot) = (q_i(\cdot), p_i(\cdot))$, because optimal control does not applies in problems with this assumption. Although this limitation is crucial to solve this kind of model using that method, we do not treat it in this paper. Indeed, we focus our attention in other limitation which raises when one use an alternative method to solve the principal’s dynamic optimization problem.

This another approach is first suggested by Mirrlees (1971). The idea is to eliminate one variable $(p_i(\cdot)$, for instance) in the optimization problem by using the indirect utility function, provided by the value function. This is made by applying the Envelope Theorem and by writing the indirect utility in an integral form. Thus, it is possible to replace this integral in the objective function. Furthermore, Fundenberg and Tirole (1995)\(^6\) show that IC constraint is equivalent to the conjunction that $q_i(\cdot)$ is nondecreasing. The principal’s optimization program then has a single variable, for each $i$, and the constraint is monotonicity of this variable.

To see how non-differentiability affects this method, let’s introduce a particular version of the Envelope Theorem applied to our setting. In order to do so we continue considering the bidimensional mechanism framework, such that the principal maximizes her utility by choosing both $q_i(\cdot)$ and $p_i(\cdot)$.

**Theorem 1 (Envelope Theorem)** Suppose that the agent’s utility function $u_i(q(\cdot), p(\cdot), \cdot)$, $u_i : X \times I \rightarrow \mathbb{R}$, for $i = 1, \ldots, k$, is continuously differentiable and that $X$ and $I$ are compact. Suppose also that the choice rule $x_i(t) = (q_i(t), p_i(t))$ is continuously differentiable. If his indirect utility function is given by $V(t) \equiv \max_{t} u_i(q(\hat{t}), p(\hat{t}), t) = u_i(q(t), p(t), t)$, then

$$\frac{dV_i}{dt} = \frac{\partial u_i}{\partial t}, \text{ for all } i. \tag{7}$$

**Proof.** Firstly, given the assumptions of compactness and differentiability, we have to note that the existence of a maximum in $u(\cdot)$, $q(\cdot)$ and $p(\cdot)$ is guaranteed by the Maximum’s Theorem. Furthermore, using the chain rule in the definition of the value function we have

$$\frac{dV_i}{dt} = \frac{\partial u_i}{\partial q} \frac{dq}{dt}(t) + \frac{\partial u_i}{\partial p} \frac{dp}{dt}(t) + \frac{\partial u_i}{\partial t}, \text{ for all } i. \tag{8}$$

Now, as the agent is a maximizer he will optimize $u_i(\cdot)$ related to his own type $\hat{t}_i$ such that $\frac{\partial u_i}{\partial q} \frac{dq}{dt}(t) + \frac{\partial u_i}{\partial p} \frac{dp}{dt}(t) = 0$. Note that maximum point is provided by the IC constraint, such that $\hat{t} = t$. The result follows immediatly of (8). □

This proof is trivial and well known but it may be instructive in our case. First, it is possible to observe the importance of the assumption of compactness in the choice set $X$ and the type set space $I$. In the case of $X$ to be unbouded, for example, utility may not have a maximum and the theorem becomes irrelevant. Violation of differentiability in the utility function also is problematic. When the partial derivatives $\frac{\partial u_i}{\partial q}$, $\frac{\partial u_i}{\partial q}$ and $\frac{\partial u_i}{\partial t}$ do not exist, the theorem does not hold. Therefore, although we allow any functional form for the agent’s utility in our generic problem above, it has to satisfy some standard assumptions.

The second point to note in the theorem’s proof is that for its results holds it is necessary that $\frac{\partial u_i}{\partial q} \frac{dq}{dt}(t) + \frac{\partial u_i}{\partial p} \frac{dp}{dt}(t) = 0$. We focus more in this assumption because we are interested in studying

\(^6\)See their theorems 8.2 and 8.3.
arbitrary mechanisms, like discontinuous ones. Moreover, in next section we argue that the violation
of compactness of $X$ makes no sense in applications. Thus, if we relax the theorem’s assumption of
differentiability, $\frac{du}{dt}(t)$ and $\frac{dv}{dt}(t)$ may not exist, what would implies that the expression (8) not hold.
Therefore, differentiability of choice rule is essential to apply the standard Envelope Theorem and
its absence makes the first step of Mirrlees’s approach fail.

If we assume to be possible to apply the Envelope Theorem, the second step of Mirrlees’s approach
concerns to write $V_i(t)$ in its integral form and replace it in the objective function. As we have
$\frac{dV_i}{dt} = \frac{\partial u_i}{\partial t}$, given by the theorem, under some conditions it is possible to integrate this expression
and to obtain
$$V_i(t) = \mathcal{V} + \int_t^1 \frac{\partial u_i}{\partial \theta}(q(\theta), p(\theta), \theta) d\theta, \text{ for all } i. \quad (9)$$

However, the conditions which allow to express the value function as an integral require more than
$\frac{dV_i}{dt} = \frac{\partial u_i}{\partial t}$ to hold. Even existence of the derivative almost everywhere, in general obtained by
monotonicity of the choice rule, is not sufficient for (9) to hold. Thus, we should impose some
restrictions in the problem in order to allow this integral representation\textsuperscript{1}. This is made in the next
section.

To summarize the problems raised by non-differentiability, we should stress the impossibility of
applying the Envelope Theorem. As the optimal control techniques also do not work under this
condition, this alternative approach to solve mechanism design models gains importance and it is
necessary to find ways of overcoming this limitation. Finally, besides non-differentiability, we have
to pay attention in the integral representation of the value function. To express it in its integral
form it is required that it has some properties, as we show in next section.

3 Generalized Envelope Theorems

Our first theorem concerns to differentiability of the value function and the form of its derivatives.
The result below generalizes the Theorem 1 of Milgrom and Segal (2002). However, our statement
is only related to differentiability, we do not focus on the left-hand and right-hand derivatives.

**Theorem 2** Suppose that $f : X \times I \subseteq \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}$ is a continuous function, with $X$ being a
compact subset of a Banach space $(\Omega, \|\cdot\|)$ and $I = \times_{i=1}^k [0, 1]$ and suppose also that $\nabla_t f : X \times I \subseteq \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^m$ is its continuous gradient. Then the value function
$$V(t) = f(x^*(t), t) = \sup_{x \in X} f(x, t), \quad (10)$$
$V : I \subseteq \mathbb{R}^m \rightarrow \mathbb{R}$, is differentiable and $\nabla_t V(t) = \nabla_t f(x^*(t), t)$, where $x^*(t) \in \text{arg max}(t)$.

**Proof.** Given $X$ is compact, then the correspondence $g : I \rightarrow \mathbb{R}^n$, $t \mapsto g(t) = X$ is compact-value
and hemi-continuous, and the Berge’s Maximum Theorem applies. Thus, $V(t) = \sup_{x \in X} f(x; t)$
is continuous in $t \in I$ and $\text{arg max}(t) = \{x \in X | f(x; t) = V(t)\}$ is upper hemi-continuous and
compact-value. Using the upper hemi-continuity of $\text{arg max}(t)$, we have that $\forall$ open set $W \subseteq \mathbb{R}^n$
\textsuperscript{1}See Royden (1968).
such that arg max(t₀) ⊆ W, there exists an open neighborhood U of t₀ such that if t ∈ U, then arg max(t) ⊆ W.

Now, let’s use the continuity of 𝜕ₓf(x, t). For given ε > 0, there exists δ such that \(\|x - x₀\| < δ\) and \(\|t - t₀\| < δ \Rightarrow \|\nablaₓf(x, t) - \nablaₓf(x₀, t₀)\| < ε\). For each x ∈ arg max(t₀) consider \(B_δ(x^*(t₀))\) e let W = \(\bigcup B_δ(x^*(t₀))\). It implies that there exists some \(δ'\) such that \(\|t - t₀\| < δ' \Rightarrow arg \ max(t) \subseteq W\).

Therefore, if \(x^*(t) \in arg \ max(t)\) so there exists \(x^*(t₀) \in arg \ max(t₀)\) such that \(\|x^*(t) - x^*(t₀)\| < δ\). Thus, if \(\|t - t₀\| < min(δ, δ')\) then \(∀x^*(t) \in arg \ max(t)\) there exists \(x^*(t₀) \in arg \ max(t₀)\) such that \(\|\nablaₓf(x^*(t), t) - \nablaₓf(x^*(t₀), t₀)\| < ε\).

Furthermore, observe that

\[
V(t) - V(t₀) = f(x^*(t), t) - f(x^*(t₀), t₀)
\]

(11)
\[
\leq f(x^*(t), t) - f(x^*(t₀), t₀)
\]

(12)
\[
\leq \nablaₓf(x^*(t), \bar{t})(t - t₀)
\]

(13)
\[
V(t) - V(t₀) - \nablaₓf(x^*(t₀), t₀)(t - t₀) \leq (\nablaₓf(x^*(t), \bar{t}) - \nablaₓf(x^*(t₀), t₀))(t - t₀),
\]

(14)

for some \(\bar{t}\) between t and t₀, where in the third inequality we use the Mean Value Theorem.

In the same way,

\[
V(t) - V(t₀) \geq f(x^*(t₀), t) - f(x^*(t₀), t₀)
\]

(15)
\[
\geq \nablaₓf(x^*(t₀), \tilde{t})(t - t₀)
\]

(16)
\[
V(t) - V(t₀) - \nablaₓf(x^*(t₀), t₀)(t - t₀) \geq (\nablaₓf(x^*(t₀), \tilde{t}) - \nablaₓf(x^*(t₀), t₀))(t - t₀),
\]

(17)

for some \(\tilde{t}\) between t and t₀.

We may combine (14) and (17) such that

\[
|V(t) - V(t₀) - \nablaₓf(x^*(t₀), t₀)(t - t₀)| ≤ \left| (\nablaₓf(x^*(t), \bar{t}) - \nablaₓf(x^*(t₀), t₀))(t - t₀) \right|
\]

(18)
\[
+ \left| (\nablaₓf(x^*(t₀), \bar{t}) - \nablaₓf(x^*(t₀), t₀))(t - t₀) \right|
\]

\[
\leq \left| \nablaₓf(x^*(t), \bar{t}) - \nablaₓf(x^*(t₀), t₀) \right| \|t - t₀\|
\]

(19)
\[
+ \left| \nablaₓf(x^*(t₀), \bar{t}) - \nablaₓf(x^*(t₀), t₀) \right| \|t - t₀\|
\]

For \(\|t - t₀\| < δ\) then

\[
\left| V(t) - V(t₀) - \nablaₓf(x^*(t₀), t₀)(t - t₀) \right| < 2\varepsilon.
\]

(21)

It implies that \(V\) is differentiable and \(\nablaₓV(t) = \nablaₓf(x^*(t), t)\).
The first point to note in the theorem above is that our assumptions are stronger than those of Milgrom and Segal (2002). Indeed, compactness of $X$ and continuity of $\nabla_t f(x(t))$ are not necessary to obtain the result for unidimensional type spaces. Therefore, we lose generality when we expand the results for multi-agent settings, in the sense of not allowing arbitrary structure for the choice set. Observe also that the theorem does not impose any assumption about the choice rule $x(\cdot)$, letting it be discontinuous, for example. However, there is a larger number of assumptions in $f(\cdot)$ than those standard models of mechanism design which respect some regularities.

It may be seen strong to impose compactness of $X$ but its necessity is justified by the use of the Berge's Maximum Theorem. Moreover, it is hard to imagine in applications any either unbounded or open choice set. In an auction model, for example, the choice set contains all the probability and payment functions. Thus, in the case of unboundedness, there are no real numbers $K_1$ and $K_2$ such that $K_1 \leq x_i(\cdot) \leq K_2$, for all $i$. Therefore, the value assumed by the choice rule $x_i(\cdot)$ goes to infinity. In our example, it means that the value paid by the bidders for the seller might be infinite (the probability function always is between 0 and 1). This would imply that the seller's utility would also increase to infinity, something extremely unreal. In all other applications a situation like that indeed would also make no sense, because Economics has this feature of its variables lie in bounded intervals\(^8\).

Similarly, to consider $X$ in a Banach space is not a strong requirement because it is just matter of including a norm.

The assumption of continuity of $\nabla_t f(x(t))$ is necessary for $\|\nabla_t f(x, t) - \nabla_t f(x^*(t_0), t_0)\| < \varepsilon$. One important point here is that the gradient is continuous both in $x(\cdot)$ and in $t$, because we need that according $x^*(t) \to x^*(t_0)$ and $t \to t_0$, so $\nabla_t f(x, t) \to \nabla_t f(x^*(t_0), t_0)$. Again, observe that all assumptions are imposed in $f(\cdot)$, letting the mechanism free to assume any structure. This is a great advantage related to other studies, like Carter (2001), for instance. This author in order to generalize Milgrom and Segal (2002)’s result requires $x(\cdot)$ continuous. Thus, his finding does not apply to discontinuous mechanisms and it is almost the same traditional Envelope Theorem. Finally, for the sake of simplicity we assume the type set as $I = \times_{i=1}^k [0, 1]$ but the generalization for $I = [a_1, b_1] \times \ldots \times [a_k, b_k]$ is direct.

Theorem 2 is only useful when the value function $V(\cdot)$ is sufficiently well-behaved, like the correspondent one in Milgrom and Segal (2002). In other words, it is necessary that it is differentiable or absolutely continuous. Therefore, it is important to identify conditions for the value function to have these properties. Next theorem does so and shows that the value function may be represented by an integral under those conditions. Yet, before presenting it, we introduce a lemma which will be useful in the theorem proof. Despite both the lemma and the theorem consider $I = \times_{i=1}^k [0, 1]$ we highlight again that it is possible to extend the analysis for a cartesian of generic ranges in the real line.

**Lemma 3** Assume the assumptions of the Theorem 2, then for any vectors $t^1, t^2 \in I = \times_{i=1}^k [0, 1]$, $t^1 \neq t^2$,

$$|V(t^1) - V(t^2)| \leq \sup_{x \in X} |f(x, t^1) - f(x, t^2)|.$$  

(22)

---

\(^8\)We do not give emphasis in the openness of $X$ because it is difficult to imagine its practical implications.
Proof. Using the definition of $V(\cdot)$,

$$V(t^1) - V(t^2) = f(x(t^1), t^1) - f(x(t^2), t^2) \leq f(x(t^1), t^1) - f(x(t^1), t^2)$$

(23)

$$\leq |f(x(t^1), t^1) - f(x(t^1), t^2)|$$

(24)

$$\leq \sup_{x\in X} |f(x, t^1) - f(x, t^2)|.$$  

(25)

In the same way,

$$V(t^2) - V(t^1) = f(x(t^2), t^2) - f(x(t^1), t^1) \leq f(x(t^2), t^2) - f(x(t^2), t^1)$$

(26)

$$\leq |f(x(t^2), t^2) - f(x(t^2), t^1)|$$

(27)

$$\leq \sup_{x\in X} |f(x, t^2) - f(x, t^1)|.$$  

(28)

Thus, as $|f(x(t^1), t^1) - f(x(t^1), t^2)| = |f(x(t^2), t^2) - f(x(t^2), t^1)|$, so the equality holds if we take the supremum, such that $\sup_{x\in X} |f(x(t^1), t^1) - f(x(t^1), t^2)| = \sup_{x\in X} |f(x(t^2), t^2) - f(x(t^2), t^1)|$, and this implies that

$$|V(t^1) - V(t^2)| \leq \sup_{x\in X} |f(x, t^1) - f(x, t^2)|.$$

(29)

■

**Theorem 4** Suppose that $f(x, \cdot)$ is absolutely continuous for all $x \in X$. Suppose also that there exists an integrable function $b : I = \times_{i=1}^{k} (0, 1) \to \mathbb{R}_+$ such that $|\partial f(x, t_1, t_2, \ldots, t_k)/\partial t_i| \leq b(t_1, t_2, \ldots, t_k)$ for all $x \in X$, all $i = 1, \ldots, k$, and almost all $(t_1, t_2, \ldots, t_k) \in I$. Then $V$ is absolutely continuous. Suppose, in addition, that $f(x, \cdot)$ is differentiable for all $x \in X$, and that $X^*(t_1, t_2, \ldots, t_k) \neq \emptyset$ almost everywhere on $I$. Then for any selection $x^*(t_1, t_2, \ldots, t_k) \in X^*(t_1, t_2, \ldots, t_k),$

$$V(t_1, t_2, \ldots, t_k) = V(0, \ldots, 0) + \sum_{i=0}^{k} \int_{0}^{t_i} \frac{\partial f}{\partial t_i}(x, s)ds.$$  

(30)

Proof. Taking $t^1 = (t_1, \ldots, t_j', \ldots, t_k)$ and $t^2 = (t_1, \ldots, t_j', \ldots, t_k)$, by the Lemma 3 we know that

$$|V(t_1, \ldots, t_j'', \ldots, t_k) - V(t_1, \ldots, t_j', \ldots, t_k)| \leq \sup_{x\in X} |f(x, t_1, \ldots, t_j', \ldots, t_k) - f(x, t_1, \ldots, t_j'', \ldots, t_k)|$$

(31)

$$\leq \sup_{x\in X} \left| \int_{t_j'}^{t_{j''}} \frac{\partial f}{\partial t_j}(x, s)ds \right|$$

(32)

$$\leq \sup_{x\in X} \int_{t_j'}^{t_{j''}} |\frac{\partial f}{\partial t_j}(x, s)| ds$$

(33)

$$\leq \int_{t_j'}^{t_{j''}} b(s)ds.$$  

(34)
This implies that $V$ is absolutely continuous. Therefore, $\frac{\partial V}{\partial t_j}$ exists almost everywhere, and we might write

$$V(t_1, \ldots, t_j, \ldots, t_k) = V(t_1, \ldots, 0, \ldots, t_k) + \int_0^{t_j} \frac{\partial V}{\partial t_j}(x, s)ds,$$  

where $0$ is in the $j$-th component of the vector $t$ in the right side of the equation. It is possible to do the same for each component $t_j$. To see this, suppose that $j = k$. Thus, we have

$$V(t_1, t_2, \ldots, t_k) = V(t_1, t_2, \ldots, t_{k-1}, 0) + \int_0^{t_k} \frac{\partial V}{\partial t_k}(x, s)ds$$

$$= V(t_1, t_2, \ldots, t_{k-2}, 0, 0) + \int_0^{t_{k-1}} \frac{\partial V}{\partial t_{k-1}}(x, s)ds + \int_0^{t_k} \frac{\partial V}{\partial t_k}(x, s)ds$$

$$\vdots$$

$$= V(t_1, t_2, \ldots, t_k) = V(0, \ldots, 0) + \sum_{i=0}^{k} \int_0^{t_i} \frac{\partial V}{\partial t_i}(x, s)ds.$$  

If $f(x, t)$ is differentiable, $\frac{\partial f}{\partial t_i}(x, s)ds$ is given by the Theorem 2 wherever it exists, then

$$V(t_1, t_2, \ldots, t_k) = V(0, \ldots, 0) + \sum_{i=0}^{k} \int_0^{t_i} \frac{\partial f}{\partial t_i}(x, s)ds.$$  

In this second theorem, we do not need to make more assumptions than those made by Milgrom and Segal (2002) in their theorem 2. Following the first theorem’s fashion, we only impose structure in $f(\cdot)$, then we continue to allow $x(\cdot)$ to be arbitrary. Besides the absolute continuity of $V(\cdot)$, this theorem also concerns to its integral representation, because the key role that (39) plays in mechanism design, according to what we have shown in the literature review.

The necessity of imposing absolute continuity\(^9\) is stressed by Milgrom and Segal (2002) in commenting the fact that value function is differentiable almost everywhere does not imply that it equals the integral of the derivative\(^10\). One possibility which is not kept out is the value function to be discontinuous. In general, the differentiability almost everywhere is obtained in applications using monotonicity of the choice rule and the integral representation uses the specific functional forms for utility to be achieved. Myerson (1981), for example, uses a linear setting.

We conclude this section reminding that our results are more general than Krisnha and Maenner (2001) and Chung and Olszewski (2007), because we do not assume any functional form for $f(\cdot)$. Although we require some technical properties in the objective function, its functional form is free to be adapted to any application. This is an advantage, because our results encompass models in all Economics’ fields.

---

\(^9\)Chung and Olszewski (2007) argue that assumption is not elegant and suggest alternative ones. See the literature review about the limitations of their suggestions.

\(^10\)The necessity of the integrable bound $b(t_1, t_2, \ldots, t_k)$ also is stressed by the authors.
4 Concluding Remarks

Non-differentiability has limited the scope of mechanism design applications. Examples of discontinuous mechanisms have raised and theory has not given a complete formal base for solving them. The main limitation imposed is the impossibility of using the Envelope Theorem in their resolution. In this context, Milgrom and Segal (2002) are an exception, because they generalize this theorem for arbitrary choice sets, allowing that mechanisms to be discontinuous. However, their findings are related only to unidimensional type spaces and may not readily be expanded to multi-agents settings.

We generalize Milgrom and Segal (2002) results for multidimensional type spaces. In order to do so, we need to use Berge’s Maximum Theorem and then impose compactness in the choice set. However, we argue that this is not a strong assumption, given it is not common in practice. We also identify conditions for the value function to be absolutely continuous and may be represented by an integral. These results allow a large number of applications in multi-agent settings. In particular, models of Public Economics, like those of public goods supply and optimal taxation, and auction theory, are suitable examples of our findings’ applications.

References

